Connecting the Firm’s Optimal Output and Input Decisions

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Abstract: The paper presents a figure and some simple numerical/algebraic examples that highlight the connection between the firm’s calculation of the profit maximizing level of output and the firm’s calculation of the profit maximizing level of input. Essentially, these two problems are one and the same except they are solved in different dimensions. Put another way, each of these problems uses the same three types of underlying information but combines them in different orders to highlight either the output market or the input market. Every intermediate microeconomics textbook treats these two problems in different chapters and in most cases the opportunity to show the connection is not exploited. As such, many students think they are attempting to learn (or memorizing) two separate models when one intuitive approach, as developed in this paper’s simple figure, will suffice.

There is an often-missed opportunity in the teaching of intermediate microeconomic theory. Invariably, the presentation of the firm’s calculation of the profit maximizing output level is covered in a different chapter than the firm’s optimal input decision, even though these two decisions are actually the same problem in two different guises. It is not improper that the topics of the profit maximizing output level and the profit maximizing input level each get their own chapter, but because of this, unfortunately, the connection between the two topics is not stressed. As such, many students remember the logic and mathematics of $MC = MR$ (marginal cost equals marginal revenue) but fail to make the connection to the same equation expressed in different variables and presented as: $VMP_L = P_L$ (value of marginal product of labor equals price of labor) or in its more general form: $MRP_L = MFC_L$ (marginal revenue product of labor equals marginal factor cost of labor). Students think they are learning (or perhaps attempting to memorize the alphabet soup of) two different models when it is really the same model, with

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the same underlying information, solved with the same technique of differential calculus, only in different dimensions.1

Adding to the students’ difficulty is the fact that, in most texts, intervening the two chapters on optimal output and input decisions are chapters on competition and price taking, monopoly, price searching, and price discrimination, and monopolistic competition and/or simple oligopoly models.2 This could be as few as two or three chapters or as many as five. Even worse, in some universities where intermediate theory is taught in a two-course sequence, the topics are covered in separate courses. The separation between the topics of profit maximization with respect to output and with respect to input is natural, which makes presenting the connection between the two all the more important. This paper presents a simple schematic diagram highlighting the necessary and sufficient information used to solve the firm’s optimization problem and showing how the information fits together.

The Intuitive Argument

Before any numeric, algebraic, or diagrammatic analysis of the firm’s profit maximization is attempted, the student should be able to recognize intuitively that three types of information are required, namely, information about how to obtain the inputs, information about the demand for the output, and information about how to turn the inputs into the output. This is pictured schematically in the rectangles in Figure 1. The relevant information about the demand

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1 Jurgen Brauer (2005) makes a similar point about principles textbooks with respect to how optimization in competitive markets leads to allocative efficiency in a multitude of settings. Each setting may seem like a different model, but the underlying principle is the same in each case and textbooks often miss the chance to stress the similarity.

2 A variety of authors and editions were sampled, including Browning and Zupan (1999), (2006); DeSerpa (1985); Frank (2000), (2006); Grinols (1994), Hirshleifer and Hirshleifer (1998), and Glazer (2005); Hyman (1988); Mathis and Koscianski (2002); Perloff (2008); Thompson and Formby (1993); and Varian (1987), (2005).
for the output, in the rightmost rectangle, is the extra revenue earned when one more unit is sold, namely, the marginal revenue (MR). The center rectangle represents the relevant information about the production conditions, that is, the extra output associated with increasing the inputs. This is called the marginal product of the input (MP). Finally, information about obtaining the inputs consists of the extra costs incurred as more inputs are used. This is called the marginal factor cost (MFC). If any of these types of information are missing, the firm’s problem simply cannot be solved. The marketing and sales departments specialize in knowing everything there is to know about the demand for the product, but they cannot possibly develop a pricing structure without knowledge of the cost of production. Similarly, production engineers know how to turn inputs into outputs, but cannot possibly decide the optimal level of either without also considering the costs of the inputs and the demand for the output. Once the student recognizes the necessity of the three types of information, and learns to distinguish among them, the problem reduces to one of how to effectively combine them.

The essence of this note is that these three necessary pieces of information can be (and are) combined in two different ways, resulting in what seems like two distinct models presented in two different chapters, but that it is really only one model. Even though the focus (optimal output versus optimal input), the nomenclature (MR and MC versus MRP and MFC) and the diagrams (demand and supply in output markets versus demand and supply in input markets) seem different, the two inquiries are essentially opposite sides of the same coin.

Now consider Figure 2, which represents the usual combination of the three types of information in the determination of the profit maximizing output level. Information about the
middle box, the production function, is usually presented first in a chapter that starts with the idea of marginal product (MP) for a single input and culminates with the presentation of the isoquant map in the two input case. The next chapter in most textbooks recognizes that the inputs cost something and adds the isocosts to the isoquants to show optimal input combinations for any output level, as traced out in the scale expansion path. From the scale expansion path

**Figure 2**
Combining the Information for Optimal Output

comes the total cost curve and from the total cost curve comes the marginal and average cost curves, and ultimately the supply curve. These represent the relevant combination of information about the cost of the inputs and about the production function as depicted in the leftmost circle in the top of Figure 2. Intuitively, one could ask, “Where does the supply curve come from?,“ and the answer must be, “From information about (and only information about) the supply of inputs and the production function.”

Meanwhile, the rightmost circle in the top of Figure 2 captures the information that is gleaned directly from (and only from) the information about the demand for the product as represented by the demand curve and the marginal revenue function. And finally, the optimal output level combines demand and supply or equates MC and MR as depicted in the triangle in Figure 2.
In a later chapter on optimal input usage, the same three types of information are combined in a different order, as shown in Figure 3. The rightmost circle combines the demand for the output with the underlying productivity of the input to get the basic VMP, the more general MRP, and ultimately the derived demand for the input. Intuitively, there is a demand for the input because there is a demand for the output and because the input can be turned into the output. The MRP is a measure of what is good about hiring more of the input and this must be compared with what is bad about hiring more of the input, namely, that one has to pay for it, as measured by the MFC, which comes directly from the information about the supply of the input and is shown in the leftmost circle in Figure 3. To complete the analysis, demand and supply or MRP and MFC, are equated as shown in the bottom triangle in Figure 3.

Figures 1-3 are combined in Figure 4, which can be used at several points in the class. When production theory is first introduced, the middle row of rectangles in Figure 4 presents an outline of how the topics will unfold, showing the place of production theory, and foreshadowing the upcoming material on input costs and demand for the output. Later, when
cost theory is developed the upper left circle in Figure 4 recaps how the cost curves come from information

Figure 4
The Full Diagram

about the production function and the supply or cost of inputs. Still later, when the revenue function is introduced, the full analysis of the firm’s optimal output decision is completed, as represented by the top triangle in Figure 4. Finally, weeks later, or as a review of a firm’s optimal output decision in a second course, Figure 4 can be reintroduced when factor demand and equilibrium in input markets are being covered. The bottom right circle shows exactly where derived demand comes from, and the bottom triangle captures equilibrium of demand and supply in input markets. There is value all along this progression, however, the main payoff
of Figure 4 is in showing how the analysis of input markets connects with the earlier material about the profit maximizing output.

One Variable Input

A simple example illustrates the above connection. Consider the case of one variable input, L, and one output, Q. We need three pieces of information. To keep the problem simple, I shall present the case of price taking behavior in both output and input markets. Therefore, the required information about the supply of the input is simply its price, denoted \( P_L \), and the required information about the demand for the output is also simply the price, \( P_Q \). The third piece of information is the production function, \( Q = f(L) \).

To find the profit maximizing level of either \( Q \) or \( L \), the same logic is used, namely, write out the profit function and differentiate with respect to the variable of interest. The profit function is always the same:

\[
\Pi = TR - TC = P_QQ - P_LL \tag{1}
\]

where \( \Pi \) is profit, \( TR \) is total revenue, and \( TC \) is total cost. Since the prices are constant it remains only to differentiate, with respect to the variable of interest, either \( Q \) or \( L \) and set the result equal to zero. If the profit maximizing quantity is desired, then \( TC \) must be written as a function of \( Q \) by simply substituting for \( L \) using the information in the production function, specifically, the inverse production function, \( L = f^{-1}(Q) \). If the profit maximizing level of input is desired, then \( TR \) must be written as a function of \( L \) by substituting for \( Q \) using the production function itself.

For example, suppose \( P_Q = 8 \), \( P_L = 2 \), and \( Q = f(L) = 5(L)^{1/2} \), (and therefore, \( L = f^{-1}(Q) = Q^2/25 \)). Then, writing (1) to be a function of \( Q \) by substituting 8 for \( P_Q \), 2 for \( P_L \), and \( Q^2/25 \) for \( L \) yields:

\[
\Pi = 8Q - 2(Q)^2/25
\]

Differentiating, setting to zero, and rearranging yields:

\[
8 = 4Q/25
\]

where the left hand side is \( MR \) and the right hand side is \( MC \). The solution for \( Q \) is 50.

Alternatively, writing (1) as a function of \( L \) (essentially what mathematicians call a change of variable technique) involves substituting for \( P_Q \) and \( P_L \) as before and substituting
5(L)^{\frac{1}{2}} for Q:

\[ II = 8(5)(L)^{\frac{1}{2}} - 2L. \]

Differentiating with respect to L, setting to zero, and rearranging yields:

\[ 20/(L)^{\frac{1}{2}} = 2, \]

where the left hand side is the MRP (also called the VMP in the price taker case) and the right hand side is the MFC (which is simply \( P_L \) in the price taker case). The solution for L is 100, which implies (and is implied by) the Q = 50 result from the first maximization.\(^3\)

In abstract form there are three necessary pieces of information. First, MR is derived from the demand for the output. Second, the marginal product of L, \( MP_L \), is derived from the production function. Third, MFC is derived from the supply of the input. These three pieces of information are combined in two different ways depending upon which one of profit maximizing input or output is sought. When Q is sought the derivative of profit with respect to Q is set to zero and arranged as:

\[ MR = \frac{MFC}{MP_L}, \]

where these functions are expressed as functions of Q, and where the right hand side is equal to marginal cost. Diagrammatically this is shown as the MR = MC intersection in Q, \( P_Q \) space.

Alternatively, when L is sought the derivative of profit with respect to L is set to zero and arranged as:

\[ MR(MP_L) = MFC_L. \]

This is the same equation, but it is rearranged and each factor in the equation is expressed as a function of L. And, of course, the left hand side gives the definition of the marginal revenue product, MRP, which simplifies to the value of marginal product, VMP, in the price takers case. A diagrammatic exposition would show an intersection of MRP and MFC in L, \( P_L \) space.

Whenever the analysis is kept to a purely diagrammatic level, there is no obvious connection between demand and supply in output space and demand and supply in input space. The abstract algebra clearly shows the connection, but is lost on the mathematically challenged. The schematic in Figure 4 goes a long way to help make the connection in a less technical, but intuitively appealing manner.

\(^3\) And note that in each case three substitutions were made into the profit function, one from the output demand, one from the input supply, and one from the production function.
Two Variable Inputs

The consideration of the two variable input case complicates matters somewhat but the connection can still be made. Suppose the production function is \( Q = f(K,L) \), where \( K \) and \( L \) are both variable inputs. The profit function now becomes:

\[
II = TR - TC = P_Q Q - P_L L - P_K K \quad .
\]  
(2)

The last two terms make up the total cost function. To simplify matters even more, assume that the firm is a price taker in the input markets so \( P_L \) and \( P_K \) are constants. The last two terms in (2) have to be rewritten as functions of \( Q \) in order to find the profit maximizing quantity. Two relationships are needed to transform the total cost from a function of \( K \) and \( L \) to a function of \( Q \). The production function supplies one such relationship. The other is provided by the scale expansion path which is derived through a constrained output maximization or cost minimization problem (diagrammatically by the tangency of isocosts and isoquants) and is written variously as:

\[
\frac{MP_L}{MP_K} = \frac{P_L}{P_K} \quad \text{or} \quad \frac{P_L}{MP_L} = \frac{P_K}{MP_K} ,
\]

where the first equation highlights the (negative of the) slopes of the isoquants and isocosts and the second highlights the requirement that \( MC \) must be equated between increases in either input.

Using the scale expansion path and the production function to eliminate \( L \) and \( K \) from (2) the profit function can be differentiated with respect to \( Q \) leading to the familiar \( MC = MR \) equation for the optimum and the related diagrammatics.

Alternatively, in the model of factor demand, the \( Q \) is eliminated from (2) using the production function. The resulting profit, as a function of \( K \) and \( L \), is maximized over the two choice variables, \( K \) and \( L \). The two resulting first-order conditions are the familiar \( MRP = MFC \) conditions, one for each input. These are solved simultaneously for \( K \) and \( L \). As part of this solution process, the factor representing the marginal revenue may be eliminated from these equations implicitly yielding the scale expansion path.

As one final alternative, if only the demand and supply for one input, say \( L \), are required, then the production function and the scale expansion path are used to eliminate both \( Q \) and \( K \) from (2) yielding profits as a function of \( L \) only. The first-order condition resulting from maximization over \( L \) can be rearranged to yield a demand equals supply intersection in \( L, P_L \) space.
A Numerical Example

To illustrate the above consider the following numerical example for the two-input case using a constant returns to scale production function, price taking in the input markets, and a downward sloping demand curve. We need three types of information:

Information about demand for output: \( P_Q = 100 - Q \),

which yields the marginal revenue: \( MR = 100 - 2Q \); 

Information about input supply: \( P_L = 25; P_K = 100 \),

which yields the marginal factor costs: \( MFC_L = 25; MFC_K = 100 \); and 

the production function: \( Q = 10(L)^{1/2}(K)^{1/2} \),

which yields the marginal products: \( MPL = 5K^{1/2}/L^{1/2}; MP_K = 5L^{1/2}/K^{1/2} \).

First, consider the problem of finding the profit maximizing output level. To write equation (2) as a function of \( Q \) only, we must express the last two terms, that is, the total cost, as a function of \( Q \). Here, as the top left circle of Figure 4 indicates, we use only the information about the input supply and the production function when we call on the optimizing condition, also known as the scale expansion path:

\[
\frac{MPL}{MP_K} = \frac{P_L}{P_K}.
\]

Plugging in for prices and marginal products and simplifying, we get:

\[
\frac{K}{L} = \frac{25}{100}
\]

or,

\[
L = 4K.
\]

Now, substituting this into the production function yields:

\[
Q = 10(4K)^{1/2}(K)^{1/2}
\]

\[
Q = 20K
\]

\[
\frac{Q}{20} = K
\]

And, successive substitution (first for prices and \( L \) and then for \( K \)), into the total cost function, yields:

\[
TC = P_LL + P_KK
\]

\[
TC = 25(4K) + 100K
\]

\[
TC = 10Q.
\]
Only now is the third piece of information about the demand curve added so that the profit function in equation (2) becomes:

\[ \Pi = P_Q Q - P_L L - P_K K \]

\[ \Pi = (100 - Q)Q - 10Q, \]

which can be differentiated with respect to \( Q \) to yield the first-order condition:

\[ 0 = 100 - 2Q - 10, \]

which is rewritten so the left hand side is \( MR \) and the right hand side is \( MC \) as:

\[ 100 - 2Q = 10. \]

The solution is \( Q = 45. \) Working backwards we find that \( K = Q/20 = 9/4, \) and \( L = 4K = 9. \)

If instead we want to find the profit maximizing levels of the inputs \( L \) and \( K, \) we rewrite equation (2) as a function of \( L \) and \( K \) only by substituting for \( P_Q \) from the demand curve, and for \( Q \) from the production function as indicated in the bottom right circle of Figure 4. Equation (2) becomes:

\[ \Pi = P_Q Q - P_L L - P_K K \]

\[ \Pi = (100 - Q)Q - 25L - 100K \]

\[ \Pi = 100Q - Q^2 - 25L - 100K \]

\[ \Pi = 100(10L^{1/2}K^{1/2} - 100L - 25L - 100K) \quad (3) \]

Equation (3) is partially differentiated with respect to \( L \) and \( K \) to derive the first-order conditions:

for \( L: \)

\[ 500L^{1/2}/K^{1/2} - 100K = 25, \]

and for \( K: \)

\[ 500L^{1/2}/K^{1/2} - 100L = 100. \]

Note that the left hand sides of these equations are the MRP’s expressed as functions of \( L \) and \( K, \)

and the right hand sides are the prices of \( L \) and \( K \) respectively. Since \( MRPL = (MR)(MP_L), \) and the same is true for \( K, \) the left hand sides are derived using only information from the demand for output and the production function as indicated in the bottom right circle in Figure 4. These two equations solve for \( L = 9 \) and \( K = 9/4, \) which imply that \( Q = 45 \) as above.

Finally, if we were interested only in the \( L \) market we could eliminate \( K \) from equation (3) by substituting in from the scale expansion path, \( K = L/4. \) Thus, equation (3) becomes:
Π = 100(10)L^{1/2}(L/4)^{1/2} - 100L(L/4) - 25L - 100L/4, or
Π = 500L - 25L^2 - 25L - 25L

and differentiating with respect to L yields the first-order condition:

0 = 500 - 50L - 25 - 25

which can be written to emphasize the demand price (the left hand side) and the supply price (the right hand side) for L as:

475 - 50L = 25

Once again, the solution is L = 9.

**Concluding Remarks**

It has become commonplace in intermediate microeconomic theory to introduce an alphabet soup of definitions and optimizing equations involving MR, MC, MFC, MP, MRP, and so on. Many students attempt to memorize the relevant conditions without an underlying understanding of their meaning. Some students will actually succeed in their endeavors. But these students are not thinking intuitively about the problem. Neither are they approaching the subject in a fashion that will allow them to extend their analysis beyond the usual bounds of the traditional textbook material.

By showing the connection between the profit maximizing level of output on the one hand and input on the other, the stage is set to consider any other decision variable that the firm may come across. The steps in the analysis are always the same. First, the profit function is written in the general form of (1) or (2). This requires information of three types as described herein. Second, the profit function must be expressed as a function of the decision variable or variables of concern. This step will require innovative thinking for a student tackling any problem that does not fit the standardized template. For actual management in the case of a real world application, this step will require painstaking attention to institutional detail and, perhaps, subjective speculation about unknown or unmeasurable effects. However, even if the exact functional relationships between a decision variable and, in turn, the demand for output, the supply of an input, or the production function are not known, the areas where more exact knowledge is required are highlighted. This provides useful guidance to the entrepreneur who must make decisions in a less than complete information setting.
For example, consider the choice of the firm to increase advertising. The decision maker can compartmentalize his or her analysis into discovering and measuring the effect of advertising on each of the three boxes in Figure 1. Advertising affects the demand for output in the rightmost box. That is obvious. Advertising also affects either or both of the other boxes. For example, if advertising, $A$, is simply an input to be purchased at price $P_A$, then the middle box is not changed but another term, $AP_A$, must be added to the total cost function in an expanded version of equation (2). Alternatively, $A$ may be produced along with $Q$ by a combination of $L$ and $K$, and thus change the formula of the production function. Providing measurements, estimates, or forecasts of these effects is the hard work of the firm’s management, but Figure 4 highlights the three separate areas in which to search out the required information, and shows how the information fits together.

The examples in this paper show how the profit maximizing output decision and the profit maximizing input decision are really the same problem using the same underlying information but combining the information in a different order and in different dimensions. The output decision first uses the production function and the input supply conditions to derive the total cost curve from which the marginal cost is derived. Then, the information about the demand is added to give the familiar $MC = MR$ condition where $Q$ is the variable. Alternatively, the input decision first combines the demand for output with the marginal products to develop the derived demands for the inputs captured in the MRP or VMP functions. Then, the information about the supply of inputs is added to give the familiar $MRP = MFC$ conditions where the inputs $L$ and/or $K$ are the variables. Figure 4 is useful in helping the teacher and student keep track of how to combine the necessary and sufficient information in a profit maximization problem.
References


